and (6.7) yields $e_2 = U^{-1}G^{-1}e$ and the vector e is defined in terms of the known thermodynamic derivatives $e_i v_*$ using the formulas (6.1).

It can easily be shown that the scalar product $e\partial q / \partial r$ is invariant under the linear transformations introduced above. From this it follows that Eq. (5.6) finally assumes the form

$$2(\varepsilon m_0 v - \varepsilon_a^2 \beta_f) \frac{\partial v}{\partial r} + \Delta \left[2\frac{\partial v}{\partial t} + (\nu - 1)\frac{v}{t}\right] = \varepsilon_a^2 \frac{p_0}{\rho_0 a_0^2} e_2 \frac{\partial q_2}{\partial r}$$

and, together with Eqs. (6.7) it forms a closed system of the order equal to the number of the relaxation processes plus one.

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ON THE APPLICATION OF CERTAIN GENERALIZATIONS OF THE AREA THEOREM IN SYSTEMS WITH ROLLING OF RIGID BODIES

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Among the attempts to extend the applicability conditions of the general theorems of dynamics, a prominent position is occupied by several generalizations of the area theorem proposed by Chaplygin and successfully applied by him to solving a number of problems on the rolling of spheres [1, 2]. Further generalizations of the area theorem appear in [3, 4]. Chaplygin's generalizations are based on the theorem on the variation of the moment of momentum relative to a moving line of fixed orientation constantly passing through some moving point [1]. We show below that in the classical problem of the rolling of rigid bodies without slippage the theorem's hypotheses completely determine the form of the surfaces of the rolling bodies.

1. We cite the most general one of the known formulations of the theorem on the variation of the moment of momentum of a mechanical system relative to a moving axis. The system consists of an arbitrary number of material points and ideal constraints can be imposed on it. Let a certain line AL retain a fixed orientation in space and pass constantly through a moving point A.

T h e o r e m 1 [5, 6]. If (1) among the possible displacements of the system there is a rotation of the whole system as a rigid body around the axis AL and (2) the condition

$$(\mathbf{v}_A \times \mathbf{v}_G, \mathbf{e}) = 0 \tag{1.1}$$

is satisfied, then the time derivative of the system's moment of momentum relative to this axis equals the sum of the moments of all forces acting on the points of the system relative to this same axis

$$\frac{d}{dt} K_{AL} = M_{AL} \tag{1.2}$$

Here $\mathbf{v}_A$ is the velocity of point A, $\mathbf{v}_G$ is the velocity of the system's center of mass, $\mathbf{e}$ is the unit vector along axis AL. Point A may not coincide with any specific material point of the system during the whole time of motion. Each time we wish to stress this circumstance we shall call it the "geometric point". It is evident that if condition (1.1) is satisfied for one point of axis AL, it is satisfied for any other point of this axis. Chaplygin's theorem [1] requires the satisfaction of the less general condition $\mathbf{v}_A = \lambda \mathbf{v}_G$ (λ is an arbitrary constant) instead of (1.1).

Let us analyze the possibility of satisfying the hypotheses of the above-stated theorem when the mechanical system is a rigid body bounded by surface S_2, which rolls on a fixed surface S_1, while the constraints express the absence of slippage at the point A of contact of surfaces S_1 and S_2. We assume that surfaces S_1 and S_2 are tangent to each other at no more than at one point and admit of a twice continuously differentiable parametrization. We consider an arbitrary instantaneous position of the body. We specify the vectors by their coordinates in some reference frame XYZ with the origin at point A. The axis AZ is directed along the common normal to the surfaces, while the axes AX and AY are directed along the curvature lines of surface S_1 ($AXYZ$ is a rectangular coordinate system). The possible displacements of the system express, obviously, a rotation of the body around point A; therefore, the condition (1) can be satisfied only if the moving axis mentioned in the theorem passes through point A. Let $\mathbf{e}$ (α, β, γ) be the unit vector along this axis AL.

The body's instantaneous angular velocity can be decomposed at point A into two components [5, 6]: the rotating angular velocity $\boldsymbol{\Omega}$ directed along the common normal to the surfaces and the angular velocity of pure rolling $\boldsymbol{\omega}$ located in the common tangent plane π. The angular velocity of the rotation of the plane tangent to surface S_i ($i = 1, 2$) at point A has the components $-k_2^{(i)} v_2$ and $k_1^{(i)} v_1$, 0, where $k_1^{(i)}$ and $k_2^{(i)}$ are the curvatures of the normal cross sections of the surface along the coordinate lines AX and AY, respectively, and $\mathbf{v}_A$ $(v_1, v_2, 0)$ is the velocity of the geometric point

A. Consequently, considering the body's pure rolling as a complex motion (the rolling of plane π over fixed surface S_1 and the rolling of surface S_2 over plane π), we obtain

$$\omega\,[-(k_2^{(1)}+k_2^{(2)})\,v_2,\ (k_1^{(1)}+k_1^{(2)})\,v_1,\ 0]$$

As surface S_2 rolls along surface S_1 let their point of contact trace out certain curved lines L_1 and L_2 on these surfaces. By k_{g1} and k_{g2} we denote the geodesic curvatures of these lines at point A. Then, obviously

$$\Omega\,[0,\ 0,\ (k_{g2}-k_{g1})\,|\mathbf{v}_A|],\quad |\mathbf{v}_A|=\sqrt{v_1^2+v_2^2}$$

It is now easy to determine the velocity components of the body's center of mass

$$\begin{aligned}\mathbf{v}_G\,[&z\,(k_1^{(1)}+k_1^{(2)})\,v_1-y\,(k_{g2}-k_{g1})\,|\mathbf{v}_A|\\ &z\,(k_2^{(1)}+k_2^{(2)})\,v_2+x\,(k_{g2}-k_{g1})\,|\mathbf{v}_A|,\\ &-y\,(k_2^{(1)}+k_2^{(2)})\,v_2-x\,(k_1^{(1)}+k_1^{(2)})\,v_1]\end{aligned}$$

where $x,\ y,\ z$ are the coordinates of the center of mass. Condition (1.1) can be written as

$$\begin{aligned}&-\alpha v_2\,[y\,(k_2^{(1)}+k_2^{(2)})\,v_2+x\,(k_1^{(1)}+k_1^{(2)})\,v_1]+\beta v_1\,[y\,(k_2^{(1)}+k_2^{(2)})\,v_2+\\ &x\,(k_1^{(1)}+k_1^{(2)})\,v_1]+\gamma\,[zv_1v_2\,(k_2^{(1)}-k_1^{(1)}+k_2^{(2)}-k_2^{(2)})+\\ &|\mathbf{v}_A|\,(k_{g2}-k_{g1})\,(xv_1+yv_2)]=0\end{aligned}\tag{1.3}$$

We note that the existence of the moving axis AL can depend upon the body's position and velocity at the initial instant. But then the area integral

$$K_{AL}=\text{const}\tag{1.4}$$

following from (1.2) when $M_{AL}=0$, is obviously a partial integral of the body's equations of motion. We do not examine this singular case here. Thus, relation (1.3) must be satisfied identically relative to the independent variables $v_1,\ v_2,\ k_{g2}-k_{g1}$ which can take arbitrary values under all possible kinematically admissible motions of the body. Consequently

$$x=0,\quad y=0,\quad \gamma\,(k_2^{(1)}-k_1^{(1)}+k_2^{(2)}-k_1^{(2)})=0\tag{1.5}$$

Since the body can be tangent to the fixed point A of the fixed surface S_1 at any point of its own surface (it is clear how to change the subsequent formulations if tangency is possible at the points of only a part of the body's surface), the relation $x=y=0$ shows that the normals to surface S_2 all intersect at one point, namely, at the body's center of mass G. Taking center G as the origin, we write the condition for the colinearity of the normal and the radius-vector $\mathbf{r}\,[x\,(u,\ v),\ y\,(u,\ v),\ z\,(u,\ v)]$ of surface S_2 ($u,\ v$ are the Gaussian coordinates of the surface)

$$\frac{y_u'z_v'-z_u'y_v'}{x}=\frac{z_u'x_v'-x_u'z_v'}{y}=\frac{x_u'y_v'-y_u'x_v'}{z}\tag{1.6}$$

Hence we obtain

$$(\mathbf{r}^2)_u'=(\mathbf{r}^2)_v'=0$$

i.e. $|\mathbf{r}|=\text{const}$. Thus, S_2 is a sphere with center at point G

Let us ascertain the form that the fixed surface S_1 can have. Two cases are possible.

First case. At point A of surface S_1

$$\gamma\neq 0$$

Then from (1.5) it follows that the principal curvatures $k_1^{(1)} = k_2^{(1)}$, i.e. point A of surface S_1 is an umbilical point. By virtue of the continuity of the normal vector and of the constancy of the orientation of axis AL, on surface S_1 we can find some neighborhood of point A at each point of which $\gamma \neq 0$, i.e. by what we proved above all points of this neighborhood are umbilical points. Consequently, they belong to a spherical surface [7].

Second case. At point A of surface S_1

$$\gamma = 0$$

a) If $\gamma = 0$ at all points of some neighborhood of point A on surface S_1, then, obviously, this surface is a part of a cylindrical surface with generatrix parallel to axis AL.

b) If in any sufficiently small neighborhood of point A on surface S_1 there are points at which $\gamma \neq 0$, then by what we have proved they are umbilical points.

By the continuity of the second quadratic form of surface S_1 the values of the principal curvatures coincide at point A. On the surface we can find some neighborhood of point A all of whose points are umbilical points. Otherwise, according to (a), any arbitrarily small neighborhood of point A on surface S_1 must contain a rectilinear segment of the generatrix. This, however, would contradict the sign-definiteness of the second quadratic form at the umbilical point A. Consequently, the certain neighborhood of point A on surface S_1 is spherical. Thus, we have proved the following statement.

Theorem 1a. As a rigid body rolls without slippage on a fixed surface the hypotheses of Theorem 1 are satisfied only when the body is a sphere with a centro-symmetric distribution of mass, **while** the fixed surface is either a spherical surface (in particular, flat) or an arbitrary cylindrical surface (in the latter case the directions of the moving axis AL and of the cylinder's generatrix must coincide).

2. Three integrals of form (1.4) enable us to reduce to quadratures the complicated problem of the rolling of an inhomogeneous symmetric sphere on a rough horizontal plane [2]. Chaplygin had pointed out some other conditions for the existence of the integrals of motion.

Theorem 2 [1]. Assume that a mechanical system consisting of an arbitrary number of material points can be separated into two individual parts (subsystems): I and II with forwardly moving parallel axes BL and CL'.

1) Let the constraints and the axis BL of subsystem I satisfy the hypotheses of Theorem 1, and in addition, let the moment of the external forces acting on the points of this subsystem (neglecting the mutual reactions of subsystems I and II), taken relative to axis BL, equal zero.

2) Let the same be true for subsystem II and axis CL'.

3) Concerning the reaction forces of subsystems I and II, assume that the sums M and M' of the moments of these forces taken relative to axes BL and CL', are in a fixed ratio

$$M : M' = \mu \tag{2.1}$$

(μ is an arbitrary constant). Then the system's equations of motion admit the first integral

$$K + \mu K' = \text{const} \tag{2.2}$$

where K and K' are the sums of the moments of momenta relative to axes BL and

CL', respectively, for the first and second subsystems.

Let us ascertain the possibility of applying this theorem to the following problem: a body (subsystem I) rolls without slippage over a fixed surface S_1 and another body (subsystem II) of bounded surface S_3 rolls, also without slippage, over the surface S_2 of the first body. For brevity we call this system a composite system with rolling. Let us consider an arbitrary instantaneous position of the bodies. According to Sect. 1, hypothesis (1) of the theorem can be satisfied only if axis BL passes through the point of contact of surfaces S_2 and S_1. We denote this point by B and we denote the point of tangency of surfaces S_3 and S_2 by A. We select a fixed rectangular coordinate system XYZ with origin at point A, and we direct the axis AZ along the normal to surfaces S_2 and S_3. Obviously, without loss of generality, we can take it that the axes BL and CL' intersect the coordinate plane AXY at points with coordinates $(x_1,\ y_1,\ 0)$ and $(x_2,\ y_2,\ 0)$, respectively.

Let us write condition (2.1) as

$$\alpha R_z\,(\mu y_2 - y_1) - \beta R_z\,(\mu x_2 - x_1) + \gamma\,[R_y\,(\mu x_2 - x_1) - R_x\,(\mu y_2 - y_1)] = 0 \tag{2.3}$$

where $\alpha,\ \beta,\ \gamma$ are the direction cosines of the parallel axes BL and CL'; $R_x,\ R_y,\ R_z$ are the components of the reaction of body I on body II at point A, and $R_z\,(R_x^2 + R_y^2) \neq 0$. The vector with components $(R_x,\ R_y,\ 0)$ is the force of friction and is always directed along the vector of relative (with respect to surface S_2) instantaneous velocity of the geometric point A. But the relative velocity of the geometric point A can have an arbitrary direction in the plane AXY. Therefore, for the satisfaction of condition (2.3) it is necessary that $\mu x_2 - x_1 = 0$ and $\mu y_2 - y_1 = 0$, i.e. point A and the moving axes BL and CL' must constantly be located in one plane. Having denoted the point of intersection of the line AB with the axis CL by C we find

$$AB = \mu AC, \quad BC = BA\,\frac{\mu - 1}{\mu}$$

(Obviously, the hypotheses of Theorem 2 are not satisfied for $\mu = 1$ and $\mu = 0$). Since the second hypothesis of Theorem 2 must be satisfied for any kinematically admissible values of the body's velocity, let us consider the case when the instantaneous velocity of body I equals zero and the velocity of body II has an arbitrary admissible value. In this case the instantaneous velocities of the geometric points A and C are related by

$$\mathbf{v}_C = \mathbf{v}_A\,\frac{\mu - 1}{\mu}$$

Therefore, condition (1.1) for subsystem II takes the same form as in Sect. 1. Consequently, by Theorem 1a surface S_3 can only be spherical, and the center of mass of body II coincides with the geometric center of the sphere. Surface S_2 can be spherical or cylindrical, but, according to condition (1) and Theorem 1a, surface S_2 is spherical in some neighborhood of point B. If we assume the existence of a continuous second quadratic form at all points of surface S_2, then, obviously, this surface can only be spherical. The center of mass of body I must be located at the center of sphere S_2. Thus, we have proved

Theorem 2a. In a composite system with rolling without slippage the hypotheses of Theorem 2 are satisfied only when each body is a symmetric sphere and the fixed surface is spherical or cylindrical.

Note. Theorem 2 admits of certain generalizations [1]. One of them relates to the case when the moving system consists of n parts reacting with one another and arranged similarly to the links of a chain with free ends. For example, on a fixed surface there rolls a hollow body inside which there rolls another hollow body and inside this there is another hollow body, and so on. Another generalization of Theorem 2 relates to the case when n parts of a mechanical system interact somewhat differently: certain $n-1$ parts (the system's satellites) react on one part of it (the system's nucleous). As an example of such a system we can take a hollow body (the nucleus) with a set of nonintersecting surfaces inside it, along each of which there rolls one body (the system's satellites). Considering the examples mentioned as composite systems with rolling it is easy to see that in the absence of slippage between the touching surfaces the statement of Theorem 2a remains in force for these generalizations of Chaplygin.

3. Integral (2. 2) characterizes the transfer of a moment from one part of the system to another. An exchange between the moment of momentum of one part of a system and the momentum of another part of it can take place under specific conditions. Let us consider a translationally moving coordinate system XYZ with origin at a point B and a line CZ' parallel to BZ, where the track of C of this line has a fixed disposition on the plane BXY.

Theorem 3 [1].

1) Assume that one part (subsystem) I of a mechanical system and the axis BZ satisfy the requirements of Theorem 1, and, in addition, let the moment of the external forces acting on the subsystem's points (neglecting the mutual reactions of subsystems I and II), taken relative to axis BZ, equal zero.

2) Let the constraints imposed on subsystem II be such that they allow translational displacements without altering the configuration in any direction perpendicular to the axis BZ and let the external forces satisfy the same restriction as for subsystem I. In this condition, as in the one preceding, the possible displacements of one subsystem are examined under the assumption that the other subsystem can be set aside and be replaced by the forces of its action on the first subsystem.

3) It is further required that the moment of the reaction forces on subsystem I from subsystem II, taken relative to axis CZ', be zero.

Then the system's equations of motion admit of the first integral

$$K_{BZ} + (\mathbf{r} \times \mathbf{Q})_{BZ} = \text{const}$$

where K_{BZ} is the moment of momentum of subsystem I, taken relative to axis BZ, $\mathbf{Q}$ is the momentum of subsystem II, $\mathbf{r}$ is the vector BC.

Note. Concerning the constraints imposed on subsystem II, it is sufficient to require that there exist possible displacements of the system as a rigid body in the direction perpendicular to plane CBZ [3]. However, for composite systems with rolling this weakening does not play any role since body II, after the replacement of its couplings with body I by reaction force, becomes free.

We have the valid

Theorem 3a. The hypotheses of Theorem 3 are not satisfied in any composite system with rolling without slippage.

Proof. We use the notation introduced in Sect. 2. According to condition (1) and

Theorem 1a, surface S_2 must be spherical. In the presence of friction at point A of contact of surfaces S_2 and S_3 the theorem's hypothesis (3) can obviously be satisfied only if the axis CZ' passes constantly through point A. Consequently, the projections of the velocities of the geometric points B and A onto plane BXY must, by hypothesis, be equal, which does not obtain in general. Theorem 3a is proved.

In conclusion we emphasize that the statements of Theorems 1a, 2a and 3a are valid when the relative velocities of the bodies at their points of contact equal zero; otherwise, the statements lose force. For example, under an appropriate choice of the moving axes the hypotheses of Theorem 3 are satisfied in the problem of the rolling of a body of arbitrary form over the absolutely smooth surface of a moving sphere [1].

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GENERALIZATION OF THE RAYLEIGH THEOREM TO GYROSCOPIC SYSTEMS

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The Rayleigh theorem on the properties of the spectrum of a linear conservative mechanical system is generalized to embrace the gyroscopic systems, i. e. to the case in which the equations of motion contain, in addition to the kinetic and potential energy matrices, an arbitrary skew-symmetric matrix of gyroscopic forces.

1. Linear gyroscopic system. We shall consider a linear gyroscopic system described by the following general expression:

$$Aq^{\cdot\cdot} + \Gamma q^{\cdot} + Cq = 0, \quad q \in R^n \tag{1.1}$$

where A is the kinetic energy matrix, C is the potential energy matrix, both A and C being symmetric $n \times n$ matrices, and Γ is a skew-symmetric matrix of the gyro-